Calculus for the Social Sciences

Course booklet

Mathematics for economics students. Contains functions, differentiation, (multivariate) optimisation, focus subjects like elasticity, and more applications.

SOWISO

About SOWISO

SOWISO offers:

- a homework, practice and **learning environment**;
- personalised **feedback** on all answer attempts;
- different **testing and assessment** tools;
- customisable **mathematics courses** with explanations, examples, and endless **randomised practice exercises**;
- an authoring tool to **create original material**;
- **learning analytics** giving detailed insight into student performance;
- **integration** with your LMS/VLE.

Our learning environment guides students along as they solve problems. When doing exercises, students can enter open answer calculations or mathematical formulas. The software will analyse their answer and provide targeted feedback and hints helping the student understand the next step in the solution process, and/or highlight any mistakes they made.

SOWISO increases student engagement and saves teachers time checking and grading!

Pricing

SOWISO partners with higher education institutions on a SAAS licensing basis.

The cost for the platform starts at € 5.50 per student per year, with an additional per student per year fee of € 7.50 per course.

A second licensing model is one in which students pay for their own license in our webshop.

Our digital courses are a fully interactive alternative for paper books and offer a personalised and adaptive learning experience that fits today's generation of students.

How are courses structured?

The courses are structured in chapters and subchapters consisting of units. The unit subjects are listed in more detail on the following pages.

Each unit consists of (at least) one theory page and one package of exercises.

Theory pages contain explanations, (randomised) examples and visualisations and (interactive) graphs.

The packages of **exercises** contain on average around 10 exercises. Each of these exercises are randomised, allowing for endless practicing, and include targeted hints and personalised feedback for the students while solving the exercises.

COURSE CONTENT

Chapter 1: Functions (23 topics)

1. *Introduction to functions (7 topics)*
    a. The notion of function
    b. Arithmetic operations for functions
    c. The range of a function
    d. Functions and graphs
    e. The notion of limit
    f. Continuity
    g. Arithmetic operations for continuity

2. *Lines and linear functions (6 topics)*
    a. Linear functions with a single unknown
    b. The general solution of a linear equation
    c. Systems of equations
    d. The equation of a line
    e. Solving systems of equations by addition
    f. Equations and lines

3. *Quadratic functions (4 topics)*
    a. Completing the square
    b. The quadratic formula
    c. Factorization
    d. Solving equations with factorization

4. *Polynomials (2 topics)*
    a. The notion of polynomial
    b. Calculating with polynomials

5. *Rational functions (1 topic)*
    a. The notion of a rational function

6. *Power functions (2 topics)*
    a. Power functions
    b. Equations of power functions

Optimization: Extreme points

Criterion for a global extremum

So far, we have only been concerned with local extrema of multivariate functions. Since global maxima of functions on $\mathbb{R}^2$ are local maxima, and similarly for minima, the local information is relevant to global optimization. For special functions, in particular, for convex or concave functions, we can even draw global conclusions.

Since global extrema of functions depend on the domain on which these functions are defined, we also need to bring the domain into the picture. For the definition of convexity of a function, we need to require that the domain of the function itself is also convex in the following sense:

Convex sets and convex functions

Let S be a domain in the x, y-plane. Then S is called **convex** if every line segment between two points of S lies entirely within S.

Let f be a function defined on a convex domain S. Then f is called **convex** if the line segment connecting any two points of the graph of f has no points below the graph. In other words, if for all u, v in S and $0 \le t \le 1$, we have

$$f(t \cdot u + (1-t) \cdot v) \le t \cdot f(u) + (1-t) \cdot f(v)$$

If $-f$ is convex on S, the f is called **concave** on S.

Examples Advanced

We are now ready to formulate a sufficient condition for a local extreme point to be a global extremum.

From local to global extremum

Let S be a convex domain.

- If f is a convex function on S, then every local minimum of f is a global minimum of f on S.
- If f is a concave function on S, then every local maximum of f is a global maximum of f on S.

Proof

In the case of a differentiable convex function on a complex domain, we can even conclude that stationary points are global extrema.

From stationary points to global extrema

Suppose that S is a convex domain and f a differentiable function on S with continuous partial derivatives.

1. If f is convex, then every stationary point of f in S is a global minimum.
2. If f is concave, then every stationary point of f in S is a global maximum.

Theory example

7. *Applications (1 topic)*
    a. Applications of functions

Chapter 2: Operations for functions (15 topics)

8. *Inverse functions (4 topics)*
    a. The notion of inverse function
    b. Injective functions
    c. Characterizing invertible functions

9. *Exponential and logarithmic functions (6 topics)*
    a. Exponential functions
    b. Properties of exponential functions
    c. Growth of an exponential function
    d. Logarithmic functions
    e. Properties of logarithms
    f. Growth of a logarithmic function

10. *New functions from old (4 topics)*
    a. Translating functions
    b. Scaling functions
    c. Symmetry of functions
    d. Composing functions

11. *Applications (1 topic)*
    a. Applications of operations for functions

Chapter 3: Introduction to differentiation (7 topics)

12. *Definition of differentiation (2 topics)*
    a. The notion of difference quotient
    b. The notion of derivative

13. *Calculating derivatives (1 topic)*
    a. Derivatives of polynomials and power functions

14. *Derivatives of exponential functions and logarithms (3 topics)*

EXERCISE EXAMPLES

Feedback

Calculate the derivative $h'(x)$ of $h(x) = 7 \cdot \sin(8 \cdot x)$.

$h'(x) = 7 \cdot \cos(8 \cdot x)$	✗	No, you may have forgotten to multiply by $g'(x)$, in which $g(x) = 8 \cdot x$.
$h'(x) = 7 \cdot \cos(56 \cdot x)$	✗	The chain rule indicates that the derivative contains $f'(g(x))$, in which $f(x) = 7 \cdot \sin(x)$ and $g(x) = 8 \cdot x$. Thus, the the argument of the cosine must be equal to $g(x)$. This is not the case in your answer.
$h'(x) = 56 \cdot \cos(8 \cdot x)$	✓	Great job

Practise example

Linear formulas and equations: Linear equations and inequalities

Linear equations

Find the unique value of x for which $8 \cdot x - 6 = -6$ is true.

Give your answer in the form $x = \dots$ and simplify as much as possible.

Hint
Remember to first subtract -6 on both sides of the equation.

$8 \cdot x = -6 - 6$	✗	No, on the right-hand side, you have subtracted 6, but you should have added it.
$8 \cdot x = -6 + 6$	✓	Eliminate each of the additions and subtractions on the right.
$x = 0$	✓	Correct answer

next > | stop | redo | ? ask question

Practise example

a. The natural exponential function and logarithm
b. Rules of calculation for exponential functions and logarithms
c. Derivatives of exponential functions and logarithms

15. *Applications (1 topic)*
    a. Applications

Chapter 4: Rules of differentiation (9 topics)

16. *Rules of computation for the derivative (6 topics)*
    a. The sum rule for differentiation
    b. The product rule for differentiation
    c. The quotient rule for differentiation
    d. The chain rule for differentiation
    e. Exponential functions and logarithmic derivatives revisited
    f. The derivative of an inverse function

17. *Applications of derivatives (3 topics)*
    a. Tangent lines revisited
    b. Approximation
    c. Elasticity

Chapter 5: Applications of differentiation (5 topics)

18. *Analysis of functions (3 topics)*
    a. Monotonicity
    b. Local minima and maxima
    c. Analysis of functions

19. *Higher derivatives (1 topic)*
    a. Higher derivatives

20. *Applications (1 topic)*
    a. Applications of differentiation

Rules of differentiation: Applications of derivatives

Elasticity

It is now clear that the derivative of a function is a measure for the absolute (instantaneous) rate of change of the function. But you can also consider the relative (proportional) rate of change. This is often used in economic analysis like: the Gross Domestic Product (GDP) of Greece increased by 0.5%. Especially in issues of pricing, we are interested in the relative (percentual) reduction in demand in relation to the relative (percentual) increase in price. In general, we define the relative change of a function as follows.

Elasticity

For a positive differentiable function $f(x)$ the **elasticity**, or **relative rate of change**, for $x > 0$ is defined as:

$$\mathrm{El}_x f(x) = f'(x) \cdot \frac{x}{f(x)}.$$

More

We will illustrate this basic economic concept of *elasticity* by means of an example.

Example

One day, a motorist pays 2.50 dollars each time he uses the ferry. The number q of times motorists use the ferry one day, can be modeled very well as $q = d(p)$, where d is the quadratic demand function

$$d(p) = 100 \cdot p^2 - 8 \cdot p + 16.$$

Here p is the price $(0 \leq p \leq 4)$.

The skipper of the ferry is considering an increase of the rate, because he hopes to increase his income. Possibly some motorists will now stay away from the ferry because of the increased costs. Therefore, the skipper would like to compare changes in the variables associated with p and q.

How can he do that?

Solution
For calculating the result of the increase in the rate by 0.25 dollars to the demand, the skipper can use the difference quotient:

$$\begin{aligned} \frac{\Delta q}{\Delta p} &= \frac{d(2.75) - d(2.50)}{0.25} \\ &= \frac{100 \cdot (2.75)^2 - 8 \cdot \cdot (2.75) + 16 - 100 \cdot (2.5)^2 - 8 \cdot \cdot (2.5) + 16}{0.25} \\ &= -275.00 \end{aligned}$$

In fact, the difference quotient that is used here is the quotient of two absolute char

Theory example

Chapter 6: Multivariate functions (9 topics)

21. *Basic notions (4 topics)*
    a. Functions of two variables
    b. Functions and relations
    c. Visualizing bivariate functions
    d. Multivariate functions

22. *Partial derivatives (4 topics)*
    a. Partial derivatives of the first order
    b. Chain rules for partial differentiation
    c. Higher partial derivatives
    d. Elasticity in two variables

23. *Applications (1 topic)*
    a. Applications of multivariate functions

Chapter 7: Optimization (7 topics)

24. *Extreme points (6 topics)*
    a. Stationary points
    b. Minimum, maximum and saddle point
    c. Criteria for extrema and saddle points
    d. Convexity and concavity
    e. Criterion for a global extremum
    f. Hessian convexity criterion

25. *Applications (1 topic)*
    a. Applications of optimization

Chapter 8: Constrained optimization (5 topics)

26. *The Lagrange multiplier method (3 topics)*
    a. Lagrange multipliers
    b. Lagrange multiplier interpretation
    c. Lagrange's theorem

27. *Sufficient conditions for optimality (2 topics)*

a. Convexity conditions for global optimality
b. Second-order conditions for local optimality

Missing something? SOWISO allows teachers to create their own content in our authoring environment.

THEORY EXAMPLE

Constrained Optimization: The Lagrange multiplier method

Lagrange multiplier interpretation

- In order to understand the *Lagrange multiplier*, we will consider the constraint optimization problem as one of a series of constraint optimization problems, parameterized by c:

Optimal value function

Let $f(x,y)$ and $g(x,y)$ be bivariate differentiable functions. For each real number c in an open interval I around 0, consider the constraint optimization problem

$$\text{max (min)} \quad f(x,y) \quad \text{subject to} \quad g(x,y)=c$$

Suppose that $[x^*(c), y^*(c), \lambda^*(c)]$ are stationary points of the Lagrangian with constraint function $g(x,y)-c$ for arbitrary c in I. Assume, moreover, that $x^*(c)$ and $y^*(c)$ are differentiable functions of c on I. Then we call

- $f^*(c)=f(x^*(c), y^*(c))$ the **optimal value function** for the constraint problem $g(x,y)=0$, and
- $\lambda^*(c)$ the corresponding **multiplier function**.

Stationary point

This gives an intuitive meaning to the Lagrange multiplier λ as the rate at which the optimal value of the objective function f changes with respect to changes in the constraint constant c. This is captured by the following result:

Multiplier function theorem

If the Lagrange multiplier λ for a constraint optimization problem with objective function $f(x,y)$ and constraint $g(x,y)=0$ has a differentiable optimal value function $f^*(c)$ on an open interval I around 0 corresponding to a stationary point $[x_0, y_0, \lambda_0]$ of the Lagrangian, then the corresponding multiplier function $\lambda^*(c)$ satisfies the equation

$$\lambda^*(c)=\frac{\mathrm{d}f^*}{\mathrm{d}c}$$

In particular, we have $\lambda_0=\lambda^*(c)=f^*_c(0)$ and, for small values of c,

Theory example

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